

WEIGHTED A PRIORI ESTIMATES FOR THE SOLUTION OF THE HOMOGENEOUS DIRICHLET PROBLEM FOR THE POWERS OF THE LAPLACIAN OPERATOR .

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ABSTRACT. Let u be a weak solution of $(-\Delta)^m u = f$ with Dirichlet boundary conditions in a smooth bounded domain $\Omega \subset \mathbb{R}^n$.

Then, the main goal of this paper is to prove the following a priori estimate:

$$\|u\|_{W_\omega^{2m,p}(\Omega)} \leq C \|f\|_{L_\omega^p(\Omega)},$$

where ω is a weight in the Muckenhoupt class A_p .

1. INTRODUCTION

We will use the standard notation for Sobolev spaces and for derivatives, namely, if α is a multi-index, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{Z}_+^n$ we denote $|\alpha| = \sum_{j=1}^n \alpha_j$, $D^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$ and

$$W^{k,p}(\Omega) = \{v \in L^p(\Omega) : D^\alpha v \in L^p(\Omega) \quad \forall |\alpha| \leq k\}.$$

For $u \in W^{k,p}(\Omega)$, its norm is given by

$$\|u\|_{W^{k,p}(\Omega)} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(\Omega)}.$$

We consider the homogeneous problem

$$(1.1) \quad \begin{cases} (-\Delta)^m u = f & \text{in } \Omega \\ \left(\frac{\partial}{\partial \nu}\right)^j u = 0 & \text{in } \partial\Omega \quad 0 \leq j \leq m-1, \end{cases}$$

where $\frac{\partial}{\partial \nu}$ is the normal derivative.

In the classic paper [1], the authors obtained a priori estimates for solutions of (1.1) for smooth domain Ω given by

$$\|u\|_{W^{2m,p}(\Omega)} \leq C \|f\|_{L^p(\Omega)}.$$

Where a key tool to prove those estimates was the Calderón-Zygmund theory for singular integral operators.

On the other hand, after the pioneering work of Muckenhoupt [7], a lot of work on continuity in weighted norms has been developed. In particular, weighted estimates for a wide class of singular integral operators has been obtained for weights in the class of Muckenhoupt A_p . Therefore, it is a natural question whether analogous weighted a priori estimates can be proved for the derivatives of solutions of elliptic equations.

For the Laplace equation ($m = 1$), it was proved in [5] that for a weight ω belonging to the Muckenhoupt class A_p

$$\|u\|_{W_{\omega}^{2,p}(\Omega)} \leq C \|f\|_{L_{\omega}^p(\Omega)}$$

on a bounded domain Ω with $\partial\Omega \in C^2$.

The goal of this paper is to extend the results of [5] for powers of the Laplacian operator with homogeneous Dirichlet boundary conditions, i.e. it is to prove that

$$(1.2) \quad \|u\|_{W_{\omega}^{2m,p}(\Omega)} \leq C \|f\|_{L_{\omega}^p(\Omega)},$$

for $\omega \in A_p$, where the constant C depends on Ω , m , n and the weight ω .

The main ideas for the proof of these estimates are similar to those given in [5]. However, non trivial technical modifications are needed because, for $m \geq 2$, the Green function is not positive in general and therefore, we cannot apply the maximum principle.

2. PRELIMINARIES

We consider the problem (1.1) in a bounded domain Ω with $\partial\Omega \in C^{6m+4}$ for $n = 2$ and $\partial\Omega \in C^{5m+2}$ for $n > 2$ (the regularity on the boundary is necessary in order to use the results of the Green function given in [6]).

The solution of (1.1) is given by

$$(2.1) \quad u(x) = \int_{\Omega} G_m(x, y) f(y) dy$$

where $G_m(x, y)$ is the Green function of the operator $(-\Delta)^m$ in Ω which can be written as

$$(2.2) \quad G_m(x, y) = \Gamma(x - y) + h(x, y)$$

where $\Gamma(x - y)$ is a fundamental solution and $h(x, y)$ satisfies

$$\begin{cases} (-\Delta_x)^m h(x, y) = 0 & x \in \Omega \\ \left(\frac{\partial}{\partial \nu}\right)^j h(x, y) = -\left(\frac{\partial}{\partial \nu}\right)^j \Gamma(x - y) & x \in \partial\Omega \quad 0 \leq j \leq m-1 \end{cases}$$

for each fixed $y \in \Omega$.

Then

$$(2.3) \quad h(x, y) = - \sum_{j=0}^{m-1} \int_{\partial\Omega} K_j(y, P) \left(\frac{\partial}{\partial \nu}\right)^j \Gamma(P - x) dS$$

where $K_j(y, P)$ are the Poisson kernels and dS denotes the surface measure on $\partial\Omega$.

We recall that any fundamental solution associated to (1.1) is smooth away from the origin and it is homogeneous of degree $2m - n$ if n is odd or if $2m < n$ and the logarithmic function appears if n is even and $2m \geq n$. However, in both cases we have the known estimates of the Green function $G_m(x, y)$ and the Poisson kernels $K_j(x, y)$. In what follows the letter C will denote a generic constant not necessarily the same at each occurrence.

$$(2.4) \quad |D_x^\alpha G_m(x, y)| \leq C \quad \text{for } |\alpha| < 2m - n,$$

$$(2.5) \quad |D_x^\alpha G_m(x, y)| \leq C \log \left(\frac{2 \operatorname{diam}(\Omega)}{|x - y|} \right) \quad \text{for } |\alpha| = 2m - n,$$

$$(2.6) \quad |D_x^\alpha G_m(x, y)| \leq C |x - y|^{2m - n - |\alpha|} \quad \text{for } |\alpha| > 2m - n,$$

$$(2.7) \quad |D_x^\alpha G_m(x, y)| \leq C \frac{1}{|x - y|^n} \min \left\{ 1, \frac{d(y)}{|x - y|} \right\}^m \quad \text{for } |\alpha| = 2m,$$

$$(2.8) \quad |K_j(x, y)| \leq C \frac{d(x)^m}{|x - y|^{n-j+m-1}} \quad \text{for } 0 \leq j \leq m-1,$$

where $d(x) := \text{dist}(x, \partial\Omega)$ (see [6] for (2.4), (2.5) and (2.6) and [4] for (2.7) and (2.8)).

3. THE ESTIMATES FOR THE DERIVATIVES OF u

In this section we state pointwise estimates for the first $2m - 1$ derivatives of the function u and a weak estimate for the $2m$ derivative. These estimates will be allow to proof the main result of this work.

Lemma 3.1. *Let $u(x)$ be solution of the problem (1.1). Then, for $|\alpha| \leq 2m - 1$ we have*

$$|D_x^\alpha u(x)| \leq C Mf(x),$$

where $Mf(x)$ is the usual Hardy- Littlewood maximal function of f .

Proof:

$$\begin{aligned} |D_x^\alpha u(x)| &\leq \int_{\Omega} |D_x^\alpha G_m(x, y)| |f(y)| dy \\ &\leq C \int_{\Omega} \frac{|f(y)|}{|x - y|^{n-1}} dy \leq C Mf(x), \end{aligned}$$

by (2.4), if $2m - n + 1 \leq |\alpha| \leq 2m - 1$ and by (2.5) and (2.6), if $|\alpha| \leq 2m - n$.

□

Proposition 3.2. *Given two measurable functions f and g in Ω , for $|\alpha| = 2m$ we have that*

$$\int_D |D_x^\alpha G_m(x, y) f(y) g(x)| dy dx \leq C \left(\int_{\Omega} Mf(x) |g(x)| dx + \int_{\Omega} Mg(y) |f(y)| dy \right),$$

where $D := \{(x, y) \in \Omega \times \Omega : |x - y| > d(x)\}$.

Proof: We write $D = D_1 \cup D_2$, where

$$D_1 = \{(x, y) \in D : d(y) \leq 2d(x)\} \quad \text{and} \quad D_2 = \{(x, y) \in D : d(y) > 2d(x)\}.$$

Then, using (2.7) we have

$$\begin{aligned} \int_D |D_x^\alpha G_m(x, y) f(y) g(x)| dy dx &\leq \int_D \frac{d(y)^m}{|x-y|^{n+m}} |f(y)| |g(x)| dy dx \\ &\leq 2^m \int_{D_1} \frac{d(x)^m}{|x-y|^{n+m}} |f(y)| |g(x)| dy dx \\ (3.1) \quad &+ \int_{D_2} \frac{d(y)^m}{|x-y|^{n+m}} |f(y)| |g(x)| dy dx = I + II. \end{aligned}$$

Calling $\Omega_k(x) = \{z \in \Omega : 2^k d(x) \leq |x-z| < 2^{k+1} d(x)\}$,

$$\begin{aligned} \int_{D_1} \frac{d(x)^m}{|x-y|^{n+m}} |f(y)| |g(x)| dy dx &\leq \int_\Omega \sum_{k=1}^\infty \int_{\Omega_k(x)} \frac{d(x)}{|x-y|^{n+1}} |f(y)| dy |g(x)| dx \\ &= \int_\Omega A(x) |g(x)| dx \end{aligned}$$

with

$$A(x) \leq \sum_{k=1}^\infty \int_{\{|x-y| < 2^{k+1} d(x)\}} \frac{d(x)}{|x-y|^{n+1}} |f(y)| dy \leq 2^n \sum_{k=1}^\infty \frac{1}{2^k} Mf(x) = 2^n Mf(x).$$

In order to estimate the term II in (3.1), we first observe that for $(x, y) \in D_2$, we have that $|x-y| \geq \frac{1}{2} d(y)$. Then

$$\begin{aligned} \int_{D_2} \frac{d(y)^m}{|x-y|^{n+m}} |f(y)| |g(x)| dy dx &\leq C \int_\Omega \sum_{k=1}^\infty \int_{\Omega_{k-1}(y)} \frac{d(y)}{|x-y|^{n+1}} |g(x)| dx |f(y)| dy \\ &= \int_\Omega B(y) |f(y)| dy \end{aligned}$$

and therefore, by the same arguments used before we have that

$$B(y) \leq 2^{n+1} Mg(y)$$

and the Proposition is proved. □

In order to see how to estimate in $\Omega \setminus D$, we consider separately the function h and Γ involved in G_m .

Proposition 3.3. *If $|\alpha| \geq 2m - n + 1$, there exists a constant C such that*

$$(3.2) \quad |D^\alpha h(x, y)| \leq C d(x)^{2m-n-|\alpha|}$$

for $|x - y| \leq d(x)$.

Proof: In view of (2.3) we must find estimates for $D_x^\alpha (\frac{\partial}{\partial \nu})^j \Gamma(P - x)$ and $K_j(y, P)$.

From the general properties of the fundamental solution $\Gamma(x - y)$ we have that

$$(3.3) \quad \left| D_x^\alpha \left(\frac{\partial}{\partial \nu} \right)^j \Gamma(P - x) \right| \leq C |P - x|^{2m-n-|\alpha|-j}$$

for $|\alpha| + j \geq 2m - n + 1$, and for $0 \leq j \leq m - 1$, by (2.8) we have that

$$(3.4) \quad |K_j(y, P)| \leq C \frac{d(y)^m}{|y - P|^{n-j+m-1}}$$

for $y \in \Omega$ and $P \in \partial\Omega$.

Then by (3.3), (3.4) and the fact that if $|x - y| \leq d(x)$ then $d(y) < 2d(x)$, we have for $|\alpha| + j \geq 2m - n + 1$

$$\begin{aligned} |D_x^\alpha h(x, y)| &\leq C \sum_{j=0}^{m-1} \int_{\partial\Omega} \frac{d(y)^m}{|y - P|^{n-1+m-j}} |P - x|^{2m-n-|\alpha|-j} dS \\ &\leq C d(x)^{2m-n-|\alpha|} \sum_{j=0}^{m-1} \int_{\partial\Omega} \frac{d(y)^{m-j}}{|y - P|^{n-1+m-j}} dS. \end{aligned}$$

In order to see that each integral is finite we write $\partial\Omega = F_1 \cup F_2$, with

$$F_1 = \{P \in \partial\Omega : |P_0 - P| > 2d(y)\} \quad \text{and} \quad F_2 = \{P \in \partial\Omega : |P_0 - P| \leq 2d(y)\},$$

where $P_0 \in \partial\Omega$ is that $|y - P_0| = d(y)$. And now, the convergence of these integrals follow in a standard way.

□

It follows from the previous Proposition that for each $x \in \Omega$ and $|\alpha| \geq 2m - n + 1$ we have that $D_x^\alpha h(x, y)$ is bounded uniformly in a neighborhood of x and so

$$(3.5) \quad D_x^\alpha \int_{\Omega} h(x, y) f(y) dy = \int_{\Omega} D_x^\alpha h(x, y) f(y) dy.$$

On the other hand, although $D_x^\alpha \Gamma$ is a singular kernel for $|\alpha| = 2m$, taking β such that $|\beta| = 2m - 1$, we have that

$$(3.6) \quad D_{x_i} \int_{\Omega} D_x^\beta \Gamma(x - y) f(y) dy = Kf(x) + c(x)f(x)$$

where c is a bounded function and K is a Calderón - Zygmund operator given by

$$(3.7) \quad Kf(x) = \lim_{\epsilon \rightarrow 0} K_\epsilon f(x), \quad \text{with} \quad K_\epsilon f(x) = \int_{|x-y|>\epsilon} D_x^\alpha \Gamma(x - y) f(y) dy.$$

Here and in what follows we consider f defined in \mathbb{R}^n extending the original f by zero.

Now we are in conditions to give the following estimate:

Theorem 3.4. *Given g a measurable function and $|\alpha| = 2m$. Then there exists a constant C depending only on n, m and Ω such that, for any $x \in \Omega$,*

$$\begin{aligned} \int_{\Omega} |D_x^\alpha u(x) g(x)| dx &\leq C \left(\int_{\Omega} \tilde{K}f(x) |g(x)| dx + \int_{\Omega} Mf(x) |g(x)| dx \right. \\ &\quad \left. + \int_{\Omega} Mg(y) |f(y)| dy + \int_{\Omega} |f(x)| |g(x)| dx \right) \end{aligned}$$

where $\tilde{K}f(x) = \sup_{\epsilon > 0} |K_\epsilon f(x)|$.

Proof: Using the representation formula for u , by (3.5), (3.6) and (3.7) we have that

$$\begin{aligned}
 D_x^\alpha u(x) &= \lim_{\epsilon \rightarrow 0} \int_{\epsilon < |x-y| \leq d(x)} D_x^\alpha \Gamma(x-y) f(y) dy + c(x)f(x) \\
 &+ \int_{|x-y| \leq d(x)} D_x^\alpha h(x,y) f(y) dy + \int_{|x-y| > d(x)} D_x^\alpha G(x,y) f(y) dy \\
 (3.8) \qquad \qquad \qquad &=: I + II + III + IV.
 \end{aligned}$$

By the results given above, for I , II and III we have pointwise estimates, and obtain (in the same way that in [5]) that

$$|I + II + III| \leq C \left(\tilde{K} f(x) + |f(x)| + M f(x) \right).$$

However, for IV we have just a weak estimate. Indeed, for the Proposition 3.2 we have

$$\int_{\Omega} |IV| |g(x)| dx \leq C \left(\int_{\Omega} M f(x) |g(x)| dx + \int_{\Omega} M g(y) |f(y)| dy \right)$$

and the Theorem is proved. □

4. MAIN RESULT

We can now state and prove our main result. First we recall the definition of the A_p class for $1 < p < \infty$. A non-negative locally integrable function ω belongs to A_p if there exists a constant C such that

$$\left(\frac{1}{|Q|} \int_Q \omega(x) dx \right) \left(\frac{1}{|Q|} \int_Q \omega(x)^{-1/(p-1)} dx \right)^{p-1} \leq C$$

for all cube $Q \subset \mathbb{R}^n$.

For any weight ω , $L_\omega^p(\Omega)$ is the space of measurable functions f defined in Ω such that

$$\|f\|_{L_\omega^p(\Omega)} = \left(\int_\Omega |f(x)|^p \omega(x) dx \right)^{1/p} < \infty$$

and $W_\omega^{k,p}(\Omega)$ is the space of functions such that

$$\|f\|_{W_\omega^{k,p}(\Omega)} = \left(\sum_{|\alpha| \leq k} \|D^\alpha f\|_{L_\omega^p(\Omega)}^p \right)^{1/p} < \infty.$$

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain such that $\partial\Omega$ is of class C^{6m+4} for $n = 2$ and $\partial\Omega$ is of class C^{5m+2} for $n \geq 2$. If $\omega \in A_p$, $f \in L_\omega^p(\Omega)$ and u a weak solution of (1.1), then there exists a constant C depending only on n , m , ω and Ω such that*

$$\|u\|_{W_\omega^{2m,p}(\Omega)} \leq C \|f\|_{L_\omega^p(\Omega)}.$$

Proof: Since M is a bounded operator in $L_\omega^p(\Omega)$, by Lemma 3.1 it follows that

$$\sum_{|\alpha| \leq 2m-1} \|D_x^\alpha u\|_{L_\omega^p(\Omega)} \leq C \|f\|_{L_\omega^p(\Omega)}.$$

Therefore, it only remains to estimate $\|D_x^\alpha u\|_{L_\omega^p(\Omega)}$ for $|\alpha| = 2m$.

Let $\omega \in A_p$ and $g(x) := (D_x^\alpha u(x))^{p-1} \omega(x)$. By Theorem 3.4 we see that

$$\begin{aligned} \int_\Omega |D_x^\alpha u(x)|^p \omega(x) dx &= \int_\Omega |D_x^\alpha u(x)| g(x) dx \\ &\leq C \left(\int_\Omega \tilde{K} f(x) |g(x)| dx + \int_\Omega M f(x) |g(x)| dx \right. \\ (4.1) \quad &\left. + \int_\Omega M g(y) |f(y)| dy + \int_\Omega |f(x)| |g(x)| dx \right). \end{aligned}$$

Since \tilde{K} and M are bounded operators in $L_\omega^p(\Omega)$, applying the Hölder inequality, it follows that

$$\begin{aligned}
 \int_{\Omega} \tilde{K}f(x) |g(x)| dx &= \int_{\Omega} \tilde{K}f(x) |g(x)| \frac{1}{\omega(x)^{1/p}} \omega(x)^{1/p} dx \\
 &\leq \left(\int_{\Omega} \tilde{K}f(x)^p \omega(x) dx \right)^{1/p} \left(\int_{\Omega} |g(x)|^q \frac{1}{\omega(x)^{q/p}} dx \right)^{1/q} \\
 (4.2) \quad &\leq \|f\|_{L_\omega^p(\Omega)} \left(\int_{\Omega} |g(x)|^q \frac{1}{\omega(x)^{q/p}} dx \right)^{1/q},
 \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

In the same way, we obtain that

$$(4.3) \quad \int_{\Omega} Mf(x) |g(x)| dx \leq \|f\|_{L_\omega^p(\Omega)} \left(\int_{\Omega} |g(x)|^q \frac{1}{\omega(x)^{q/p}} dx \right)^{1/q}$$

and

$$(4.4) \quad \int_{\Omega} |f(x)| |g(x)| dx \leq \|f\|_{L_\omega^p(\Omega)} \left(\int_{\Omega} |g(x)|^q \frac{1}{\omega(x)^{q/p}} dx \right)^{1/q}.$$

For the last term in (4.1), taking into account that $\omega^{-q/p} \in A_q$, we have that

$$\begin{aligned}
 (4.5) \quad \int_{\Omega} Mg(y) |f(y)| dy &\leq \|f\|_{L_\omega^p(\Omega)} \left(\int_{\Omega} Mg(y)^q \frac{1}{\omega(y)^{q/p}} dy \right)^{1/q} \\
 &\leq \|f\|_{L_\omega^p(\Omega)} \left(\int_{\Omega} |g(x)|^q \frac{1}{\omega(x)^{q/p}} dx \right)^{1/q}.
 \end{aligned}$$

Then, by (4.2), (4.3), (4.4) and (4.5) we have

$$\|D_x^\alpha u\|_{L_\omega^p(\Omega)}^p \leq C \|f\|_{L_\omega^p(\Omega)} \left(\int_{\Omega} |g(x)|^q \frac{1}{\omega(x)^{q/p}} dx \right)^{1/q}.$$

By the definition of $g(x)$,

$$\begin{aligned}
 \left(\int_{\Omega} |g(x)|^q \frac{1}{\omega(x)^{q/p}} dx \right)^{1/q} &= \left(\int_{\Omega} |D_x^\alpha u|^{(p-1)q} \omega(x)^q \frac{1}{\omega(x)^{q/p}} dx \right)^{1/q} \\
 &= \left(\int_{\Omega} |D_x^\alpha u|^p \omega(x) dx \right)^{1/q} = \|D_x^\alpha u\|_{L_\omega^p(\Omega)}^{p/q}.
 \end{aligned}$$

Then we obtain

$$(4.6) \quad \|D^\alpha u\|_{L_\omega^p(\Omega)}^p \leq C \|f\|_{L_\omega^p(\Omega)} \|D^\alpha u\|_{L_\omega^p(\Omega)}^{p/q}$$

and the Theorem is proved for $u \in W_\omega^{2m,p}(\Omega)$.

Finally, we will show that the weak solution u of (1.1) belong to $W_\omega^{2m,p}(\Omega)$:

We have that $(-\Delta)^m u = f$, with $f \in L_\omega^p(\Omega)$, then there exists a sequence $f_k \in C^\infty(\mathbb{R}^n)$ such that $\lim_{k \rightarrow \infty} f_k = f$ in $L_\omega^p(\Omega)$ [3].

For each k , there exists $u_k \in C^\infty(\Omega)$ satisfying

$$\begin{cases} (-\Delta)^m u_k = f_k & \text{in } \Omega \\ \left(\frac{\partial}{\partial \nu}\right)^j u_k = 0 & \text{in } \partial\Omega \quad 0 \leq j \leq m-1. \end{cases}$$

It is easily to see, from Lemma 3.1 that $u_k \in W_\omega^{2m-1,p}(\Omega)$, and obviously $u_k \in W_{\omega,loc}^{2m,p}(\Omega)$. Moreover for all compact set $K \subset \Omega$, we have

$$\|u_k\|_{W_\omega^{2m,p}(K)} \leq C(K),$$

where $C(K)$ is a constant depending on the measure of K . Indeed, taking $v_k = u_k \varphi$ with $\varphi \in C_0^\infty(K)$, it follows that $v_k \in W_\omega^{2m,p}(\Omega)$, satisfies (1.1) with $f = g_k \in L_\omega^p(\Omega)$, and we can use (4.6).

Then, it follows by the dominated convergence theorem that $u_k \in W_\omega^{2m,p}(\Omega)$ and applying (4.6), we have that

$$\|u_k\|_{W_\omega^{2m,p}(\Omega)} \leq C \|f_k\|_{L_\omega^p(\Omega)}.$$

Therefore, $\{u_k\}$ is a Cauchy sequence in $W_\omega^{2m,p}(\Omega)$ and there exists $v \in W_\omega^{2m,p}(\Omega)$ such that $\lim_{k \rightarrow \infty} u_k = v$ in $W_\omega^{2m,p}(\Omega)$. Let see now that v solves (1.1).

Obviously, $f = \lim_{k \rightarrow \infty} f_k = \lim_{k \rightarrow \infty} (-\Delta)^m u_k = (-\Delta)^m v$ in $L_\omega^p(\Omega)$ and by the classical trace theorems in Sobolev spaces and the definition of $\omega \in A_p$, it follows that v satisfies the homogeneous boundary conditions and by uniqueness of the solution, the Theorem is proved.

□

Remark 4.2. The result of Theorem 4.1 is valid also for u a weak solution of

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ \mathcal{B}_j u = 0 & \text{in } \partial\Omega \quad 0 \leq j \leq m-1 \end{cases}$$

when $\mathcal{L} := \sum_{|\alpha| \leq 2m} a_\alpha D^\alpha$ is uniformly elliptic and $\mathcal{B}_j := \sum_{|\alpha| \leq j} b_\alpha D^\alpha$, $0 \leq j \leq m-1$ are the boundary operators defined in [1].

Indeed, we define $l_1 > \max_j(2m-j)$ and $l_0 = \max_j(2m-j)$. If the coefficients $a_\alpha \in C^{l_1+1}(\overline{\Omega})$, $b_j \in C^{l_1+1}(\partial\Omega)$ and $\partial\Omega \in C^{l_1+2m+1}$ we have that the Green function G_m and the Poisson kernels K_j for $0 \leq j \leq m-1$ exist whenever $l_1 > 2(l_0+1)$ for $n=2$ and $l_1 > \frac{3}{2}l_0$ for $n \geq 3$.

Moreover, wherever they are defined, the Green function and the Poisson kernels of the operator \mathcal{L} with these boundary conditions satisfy the estimates (2.4), (2.5), (2.6), (2.7) and (2.8) (see [4] and [6]).

Remark 4.3. Using the fact that $d(x)^\beta \in A_p$ for $-1 < \beta < p-1$ and some imbedding Theorems for weighted Sobolev spaces (see [5]) we have as a consequence of the main result

Theorem 4.4. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain as above, $f \in L_{d^\gamma}^p(\Omega)$, with $\gamma = k\beta$, where $k \in \mathbb{N}$ and $0 \leq \beta \leq 1$. If u be the solution of problem (1.1), $0 \leq \gamma < p-1$ and $\frac{1}{p} - \frac{1}{q} \leq \frac{2m}{n+k}$ (with $q < \infty$ when $2mp = n+k$), then there exists a constant C depending only on γ , p , q , n and Ω such that

$$(4.7) \quad \|u\|_{L_{d^\gamma}^q(\Omega)} \leq C \|f\|_{L_{d^\gamma}^p(\Omega)}.$$

Finally, as a particular case of (4.7) taking $\gamma = m$ we have that

$$\|u\|_{L_{d^m}^q(\Omega)} \leq C \|f\|_{L_{d^m}^p(\Omega)}$$

for $p > m + 1$ and $\frac{1}{p} - \frac{1}{q} \leq \frac{2m}{n+1}$ (with $q < \infty$ when $2mp = n + m$).

This result is proved in [4] using different arguments for the case $\frac{1}{p} - \frac{1}{q} < \frac{2m}{n+m}$.

Our results shows that, at least in the case $p > m + 1$, the estimate remains valid

when $\frac{1}{p} - \frac{1}{q} = \frac{2m}{n+m}$.

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